Passive scalar advected by a rapidly changing random velocity field: Probability density of scalar differences

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The problem of a passive scalar advected by a correlated-in-time Gaussian random velocity field is considered. The equation for the probability density $P(\Delta T,r)$ of the scalar differences is derived. This equation resembles the nonlinear Boltzmann equation reflecting the fact that the scalar fluctuations carried by the velocity field along adjacent trajectories can interact due to the action of finite diffusivity. It is shown that the probability density of the scalar differences has algebraic tails in the scale-invariant range and that the moments $S_n=\langle(\Delta T)^n\rangle$ with $n<p$ obey "normal scaling" $S_n\propto r^{n\xi_1}$ with ξ_1=const, while all moments with $n>p$ behave as $S_{n>p}\propto r^{\xi}$, where both ξ and p depend on parameters of the problem. [S1063-651X(96)12608-8]

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I. INTRODUCTION

The recent surge in interest in the problem of a passive scalar advected by a correlated-in-time random velocity field has been motivated by Kraichnan's suggestion that the high-order moments of the scalar differences (structure functions), governed by this equation, scale as $S_{2n}\equiv\langle(\Delta T)^{2n}\rangle\propto r^{\xi_n}$ with the exponents $\xi_{2n}\propto\sqrt{2n}$ when n is large enough [1]. This phenomenon is called "anomalous scaling" contrary to the "normal scaling" $\xi_n=n\xi_0$, where ξ_0=const, equal to the scaling exponent of one of the low-order structure functions. Kraichnan considered the equation for a passive scalar,

$$\frac{\partial T(x)}{\partial t}+v_i(x)\,\frac{\partial T(x)}{\partial x_i}=f(x,t)+\kappa\,\frac{\partial^2 T(x)}{\partial x_i^2}, \tag{1}$$

where decorrelated-in-time incompressible velocity field $\nabla_i v_i=0$ is defined by the correlation function

$$\overline{v_i(x,t)v_j(x,t')}-\overline{v_i(x,t)v_j(x',t')}\propto r^{\eta}\delta(t-t') \tag{2}$$

and the forcing function ($\langle f^2\rangle=1$) acts at the large scales only, so that its structure function $\phi(r)=\overline{[f(x+r)-f(x)]^2}=O((r/L)^2)$ and is negligibly small in the inertial range where $r/L\ll1$. We denote $b(r)=\langle v^2-v(x)v(x')\rangle=r^{\eta}$, set $L=1$, and consider $0<\eta\leqslant2$ in what follows. The equations for the moments S_{2n} are derived from (1) with the result:

$$\frac{\partial S_{2n}}{\partial t}-\frac{2}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1+\eta}\frac{\partial S_{2n}}{\partial r}+J_{2n}-2n(2n-1)\times\left(\frac{r}{L}\right)^2 S_{2n-2}, \tag{3}$$

where

$$J_{2n}=2n\kappa\langle[T(2)-T(1)]^{2n-1}(\nabla_1^2+\nabla_2^2)[T(2)-T(1)]\rangle. \tag{4}$$

To close the system of equations one has to calculate the correlation function (4), which is a very difficult task. Based on the symmetries of the problem and some qualitative considerations, Kraichnan postulated an expression:

$$J_{2n}=2n\,\frac{S_{2n}(r)A(r)}{S_2(r)}, \tag{5}$$

where

$$A(r)=\kappa(\nabla^2 S_2(0)-\nabla^2 S_2(r)).$$

Then, setting $r/L=0$ in the inertial range and $A(r)=\kappa\nabla^2 S_2(0)=-1$ gives

$$\frac{\partial S_{2n}}{\partial t}-\frac{2}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1+n}\frac{\partial S_{2n}}{\partial r}=-2n\,\frac{S_{2n}}{S_2}. \tag{6}$$

The most important feature of (6) is that it is a homogeneous equation and one can calculate the scaling exponents ξ_{2n} by simple power counting. Assuming that

$$S_{2n}(r)=A_{2n}r^{\xi_{2n}} \tag{7}$$

gives Kraichnan's scaling exponents:

$$\xi_{2n}=\tfrac{1}{2}\sqrt{4nd\xi_2+(d-\xi_2)^2}-\tfrac{1}{2}(d-\xi_2). \tag{8}$$

In the limit $n\gg1$ we have $\xi_{2n}=O(\sqrt{n})$. When $n\to0$

$$\xi_{2n}\approx\frac{d\xi_2 n}{d-\xi_2}\neq n\xi_2\quad(\xi_2=2-\eta).$$

Thus, the model does not generate scalar fluctuations with the structure functions $S_n(r)$ obeying "normal" scaling even when $n\ll1$. For example, when $\eta=4/3$, $\xi_2=2/3$, and $d=3$, the expression (8) gives $\xi_1/\xi_2\approx0.55$. When $\xi_2\to2$ this ratio tends to $1/\sqrt{2}$. This property of Kraichnan's is striking since, if correct, it makes the system governed by (1) very different from anything encountered in real life and numerical flows, where normal scaling for some of the low-order moments and anomalous scaling for the high-order ones is usually reported and the observed ratio $\xi_2/\xi_1=2$ with a very good accuracy.

Let us show that none of the Kraichnan exponents (8) can be derived perturbatively in terms of powers of deviations from a scaling solution. It is easy to see [2] that Kraichnan's suggestion corresponds to the following equation for the probability density function (PDF) $P(\Delta T,r)$:

$$\frac{\partial P}{\partial t}-\frac{2}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1+\eta}\frac{\partial P}{\partial r}=-\frac{\partial}{\partial \Delta T}H(\Delta T,r)P(\Delta T,r)+\left(\frac{r}{L}\right)^2\frac{\partial^2}{\partial(\Delta T)^2}P(\Delta T,r), \tag{9}$$

where $H(\Delta T,r)$ is the conditional expectation value of the operator

$$\kappa\left(\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_1^2}\right)\Delta T$$

for fixed ΔT. Kraichnan's closure is obtained from (9) neglecting the $O((r/L)^2)$ contribution and postulating

$$H(\Delta T,r)=-d\xi_2\Delta T/S_2(r). \tag{10}$$

In this expression we have set $\kappa\nabla^2 S_2(0)=-1$ so that in what follows $S_2(r)=r^{\xi_2}$.

We seek the zero-order solution of the equation of motion in a scaling form:

$$P(\Delta T,r)=\frac{1}{r^{\alpha}}F(x), \tag{11}$$

where $x=\Delta T/r^{\alpha}$ with $\alpha=\xi_2/2>0$. The resulting equation for $F(x)$ is

$$2\alpha^2x^2F'(x)+AxF(x)=0, \tag{11a}$$

where $A=2\alpha^2-2(d-\xi_2)\alpha+d\xi_2$. The constant in the right side of (11) is chosen to be equal to zero since $F(x)=F(-x)$. The solution to this equation is

$$F(x)\propto x^{-\mu}, \tag{12}$$

where $\mu=A/2\alpha^2$. In a case, treated in the numerical experiments by Chen [3], $d=2$, $\xi_2=1/2$, $\alpha=\xi_1\approx 1/4$, $A=3/8$, and $\mu=3$. We see that Kraichnan's theory does not admit scaling solution (11) and thus, no perturbative treatment in terms of deviation from ''normal'' scaling is possible. The $O((r/L)^2)$ contribution, neglected in the derivation of (11) regularizes the theory leading to the anomalous scaling. It is difficult, however, to explain such a strong direct influence of the large-scale rapidly changing in time source on the inertial range
($r/L\rightarrow 0$) dynamics of the scalar.

The purpose of this work is to present an alternative equation of motion for the probability density of the scalar differences, leading to the scale-invariant PDF but still giving anomalous scaling of the high-order structure functions. We will explore some of the ideas which proved to be extremely fruitful in investigation of the dynamics of the fluctuations generated by the random-force-driven Burgers equation. For this problem a satisfactory dynamic theory has recently been developed [4–7]. There it has been shown that the probability density of the velocity difference $\Delta u=u(2)-u(1)$ consists of two contributions: $P_s(\Delta u,r)=1/r^{\alpha}F(\Delta u/r^{\alpha})$, which is responsible for the ''normal scaling of the low-order moments,'' and another one, $P_a=r\varphi(\Delta u/u_{rms})$ dominated by the contribution from coherent structures giving rise to the anomalous scaling. It is important that u_{rms} is the r-independent value of the rms single-point velocity fluctuations. The tails of the PDF are dominated by P_a.

II. EQUATIONS FOR THE PROBABILITY DENSITY

Let us consider the generating function

$$Z_-=\langle e^{\lambda[T(2)-T(1)]}\rangle\equiv\langle e^{\lambda T_-}\rangle. \tag{13}$$

The equation of motion is

$$\frac{\partial Z_-}{\partial t}-\frac{2}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1+\eta}\frac{\partial Z_-}{\partial r}=J_-+\lambda^2\left(\frac{r}{L}\right)^2Z_- \tag{14}$$

and

$$J_-=\lambda\kappa\langle[\nabla_2^2T(2)-\nabla_1^2T(1)]e^{\lambda T_-}\rangle, \tag{15}$$

where $\nabla_i=\partial/\partial x_i$ and $T(i)=T(x_i)$. It will be useful for what follows to introduce

$$Z_+=\langle e^{\lambda[T(2)+T(1)]}\rangle\equiv\langle e^{\lambda T_+}\rangle, \tag{16}$$

which obeys the following equation of motion:

$$\frac{\partial Z_+}{\partial t}-\frac{2}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1+\eta}\frac{\partial Z_+}{\partial r}=J_++2\lambda^2\left[1-\left(\frac{r}{L}\right)^2\right]Z_+, \tag{17}$$

where

$$J_+=\lambda\kappa\langle[\nabla_2^2T(2)+\nabla_1^2T(1)]e^{\lambda T_+}\rangle. \tag{18}$$

The main difficulty is in the evaluation of the correlation function J_- since it involves calculation of the infinite chain of the correlation functions. Indeed, simple integration by parts in the relation (15) gives

$$J_-=-\lambda^2\langle[\epsilon_T(1)+\epsilon_T(2)]e^{\lambda T_-}\rangle. \tag{19}$$

We see that the equation for the PDF of the scalar difference is not closed since we have to calculate the correlation function involving the dissipation rates ϵ_T. Substituting the equation for the scalar variance into (19) we have to face the difficulty of evaluating the correlation functions involving $T^2(2)-T^2(1)=[T(2)-T(1)][T(2)+T(1)]$ and thus, as in the problem of the Burgers equation, we have to deal with correlation of $T_-=T(2)-T(1)$ with $T_+=T(2)+T(1)$. The origin of this difficulty can be traced to the fact that the probability density of the scalar difference must be obtained from the general two-point correlation function given by

$$Z_{12}=\langle e^{\lambda_1T(1)+\lambda_2T(2)}\rangle=\langle e^{\lambda T_-+\Lambda T_+}\rangle \tag{20}$$

in the limit $r\rightarrow 0$, where $\lambda=\frac{1}{2}(\lambda_2-\lambda_1)$ and $\Lambda=\frac{1}{2}(\lambda_2+\lambda_1)$. It will be shown below that the dissipation terms couple T_+ and T_- and that is why the calculation of J_- and J_+ is not a simple matter.

Before we proceed further, let us discuss some consequences of the relations presented above. First of all, we have in a statistically steady state:

$$\frac{2}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1+\eta}\frac{\partial S_n}{\partial r}=n(n-1)\langle[\epsilon_T(1)+\epsilon_T(2)]T_-^{n-2}\rangle. \tag{21}$$

The equation for Z_{12} is

$$\frac{\partial Z_{12}}{\partial t}-\frac{2}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1+\eta}\frac{\partial Z_{12}}{\partial r}=J_{12}+\lambda_i\lambda_j\overline{f(i)f(j)}Z_{12}, \tag{22}$$

where

$$J_{12}=\kappa\langle[\lambda_2\nabla_2^2T(2)+\lambda_1\nabla_1^2T(1)]e^{\lambda T_-+\Lambda T_+}\rangle$$

or

$$J_{12}=\kappa\langle\{\lambda[\nabla_2^2T(2)-\nabla_1^2T(1)]+\Lambda[\nabla_2^2T(2)+\nabla_1^2T(1)]\}e^{\lambda T_-+\Lambda T_+}\rangle. \tag{23}$$

We can see that the generating function Z_{12} can be represented as

$$Z_{12}=Z_-Z_+, \tag{24}$$

provided

$$-4r^n\frac{\partial Z_+}{\partial r}\frac{\partial Z_-}{\partial r}=J_{12}-Z_-J_+(\Lambda)-Z_+J_-(\lambda) \tag{25}$$

with J_{12} defined by (23). In general this assumption is incorrect.

Now, following Polyakov [4], we assume that in the scale-invariant regime when $r\to 0$ and $T_-\ll T_{rms}$ the Fourier-Laplace transform of the generating function $Z_{12}(\lambda,\Lambda)$ is dominated by the range of $\Lambda=O(1/T_{rms})\ll 1/T_-=O(\lambda)$. This means that only $\Lambda\to 0$ contributes to Z_{12}, or, in other words [4],

$$Z_{12}\propto\delta(\Lambda).$$

The probability density in this limit ($T_-\ll T_{rms}$) can be written as [4]

$$P(T(1),T(2))=P\left(\frac{T_-}{r^\alpha}\right)W\left(\frac{T_+}{T_{rms}}\right).$$

This relation is equivalent to (24) and it is clear that it is valid only when T_- is small. When $T_-\geqslant T_{rms}$ we can see from the equation of motion that the variables cannot be separated and the probability density $P(T(1)-T(2))$ is not invariant under the transformation $T\to T+C$ since its tails are dominated by the single-point PDF $P(T)$. In other words, the tails of the probability density $P(T_-)\propto\varphi(T_-/T_{rms})$. The importance of this very basic fact was noticed by Polyakov in the theory of the Burgers turbulence [4].

In this limit the generating function

$$Z_+=\langle e^{2\Lambda T(1)}e^{\Lambda T_-}\rangle$$

leads to

$$P(T_+,r)\approx[1+\psi(r)]P(T/T_{rms}),$$

where $P(T/T_{rms})$ is the single-point probability density and in the scale-invariant range $\psi(r)\propto(r/L)^{\xi_+}$ with $\xi_+=\xi_2$ [$\psi(r)\to 0$] when $r\to 0$. This assumption reflects the fact that in the limit T_-0 and $r\to 0$ the PDF $P(T_+,r)\to P(T)$. All this gives

$$Z_+\approx[1+\psi(r/L)]Z_s(\Lambda T)$$

and

$$Z_s=\langle e^{2\Lambda T}\rangle.$$

Equations (14) and (15) must follow (23) in the limit $r\to 0$. We assume that

$$J_{12}=[Z_-J_+(\Lambda)+Z_+J_-(\lambda)]\Psi(Z_-,Z_+),$$

where the functional $\Psi(Z_+,Z_-)$ is to be determined from the theory. Substituting this expression into (25) we find

$$J_-(\lambda)=\frac{-4r^n\frac{\partial Z_+}{\partial r}\frac{\partial Z_-}{\partial r}}{Z_+[\Psi(Z_+,Z_-)-1]}-Z_-\frac{J_+(\Lambda)}{Z_+(\Lambda)}.$$

Using the approximate relations for Z_+, introduced above, we can see that this expression defines a functional $J_-(\lambda T_-,Z_-)$, calculation of which is, in general, a difficult task since we have to know the function Ψ. However, luckily, one can arrive in the nontrivial conclusions, assuming that $J_-(Z_-)$ can be represented as a series:

$$J_-=\lambda^2(c_1Z_-+bZ_-^2+dZ_-^3+\cdots)+\lambda af(r)\frac{\partial}{\partial\lambda}Z_-, \tag{26}$$

which corresponds to the anzatz ($\Lambda\to 0$):

$$\Psi(Z_-,Z_+)\propto\frac{\Theta(Z_-)}{r^n\psi'(r/L)Z_+}+1.$$

The expression for $\Theta(Z_-)$ is found readily from (26).

The last contribution to (26), which appears in the lowest order of the expansion in powers of small T_-, was added due to the following considerations. Take a simple linear equation:

$$\frac{\partial T(x)}{\partial t}=f(x,t)+\kappa\frac{\partial^2T(x)}{\partial x_i^2}.$$

The exact steady-state solution of this equation, obeying Gaussian statistics of the forcing function, can be written down immediately. The equation of motion for the generating function of the scalar difference, corresponding to this equation is

$$\frac{\partial Z_-}{\partial t}=J_-+\lambda^2\left(\frac{r}{L}\right)^2Z_-.$$

Since we know the solution of the problem, the expression for J_- is found readily and

$$\lambda^2\langle(\Delta f)^2\rangle Z_-+\frac{\langle(\Delta f)^2\rangle}{S_2(r)}\lambda\frac{\partial}{\partial\lambda}Z_-=0.$$

ÅThe expression for J_-, defined by this equation, corresponds to the exact solution of a diffusion equation with Gaussian source in the right side. The linear equation of an advected passive scalar can be represented in terms of the effective (''eddy'') diffusivity easily extracted from (3). The solution of a problem would be a simple matter, if only the contributions coming from ''molecular diffusivity'' κ could be neglected. As we saw above this is not the case and the J_- contribution, given by (26), is vitally important. This is because the dynamics of a passive scalar consists of two competing processes. One is a mere kinematic advection of a scalar which does not lead to the decay of its variance. This process is accurately described by the effective diffusivity. The expansion in powers of Z_-, introduced above, accounts for an additional physical phenomenon, caused by the molecular diffusivity, which means that we are looking for the solution in terms of deviations from the zero-molecular-diffusivity trivial theory. The physical meaning of this expansion becomes more clear if we write the equation for the probability density $P(T_-,r)$. In the units in which $S_2(r)=r^{\xi_2}$, where $\xi_2=2-\eta$ the equation of motion is

$$\frac{\partial P}{\partial t}-\frac{2}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1+\eta}\frac{\partial P}{\partial r}=ad\xi_2\frac{\partial}{\partial T_-}\frac{T_-}{S_2(r)}P(T_-,r)+c_1d\xi_2\frac{\partial^2}{\partial T_-^2}P(T_-,r)+bd\xi_2\frac{\partial^2}{\partial T_-^2}\int P(y,r)\times P(y-T_-,r)dy+\cdots. \tag{27}$$

Now we can see that this equation is simply the Boltzmann equation for the probability density $P(T_-,r)$ taking into account binary, triple, etc., collisions between the fluctuations of the passive scalar. In the zero-diffusivity limit, when the scalar is merely carried around by the incompressible velocity field, the particles from the adjacent trajectories do not interact. Finite diffusivity, considered in this work, introduces the ''defects'' allowing migration of the scalar between different trajectories, thus leading to transitions $(T_-,r)\to(T'_-,r)$ described by the collision term in (27). In what follows we restrict ourselves to binary collisions only and will show below that accounting for the multiple collision contributions does not change the shape of the tails of the PDF. It is interesting that this equation preserves all conservation laws and the collision integral is represented in terms of the nonlinear diffusivity in the phase space. Our goal is to solve the more difficult equation (14), (26) and find all coefficients a, b, c, etc.

III. SOLUTION OF EQUATION OF MOTION

To analyze this equation let us notice that when T_- is large enough, the collision integral can be rewritten using an approximation:

$$C=\int P(y,r)P(y-T_-,r)dy\approx\int' P(y,r)P(y-T_-,r)dy+P(T_-,r),$$

where $\int'$ is the integral over the range $-T_0<T<T_0$ giving the dominant contribution to the normalization integral:

$$\int P(x)dx=1.$$

It will be shown below that near the origin the PDF $P(T_-)$ is a broad function of T_-, so that we approximate the first contribution to the right side of the equation for collision integral C as

$$\int' P(y,r)P(y-T_-,r)dy\approx f(r)P^2(T_-),$$

where $f(r)$ is easily found for the scale-invariant solutions (see below). With these approximations we have the differential equation for the PDF:

$$\frac{\partial P}{\partial t}-\frac{2}{r^{d-1}}\frac{\partial}{\partial r}r^{d-1+\eta}\frac{\partial P(T_-,r)}{\partial r}=ad\xi_2\frac{\partial}{\partial T_-}\frac{T_-}{S_2(r)}P(T_-,r)+cd\xi_2\frac{\partial^2}{\partial T_-^2}P(T_-,r)+bd\xi_2f(r)\frac{\partial^2}{\partial T_-^2}P^2(T_-,r), \tag{28}$$

where $c=c_1+b$. Let us discuss the interesting cases $\eta\to0$ and $d\to\infty$ corresponding to the Gaussian passive scalar dynamics. When $\eta=0$ the only possible velocity field is $\mathbf{v}=\mathbf{0}$ and (1) and (2) describe the diffusivity-dominated linear diffusion problem. When $d\to\infty$, Eq. (1) generates the scalar fluctuations corresponding to the weak-coupling Gaussian limit. In principle, we have to have $f=f(r,\eta,d)$ such that $f(r,\eta,d)\to0$ in the limits $\eta\to0$ or $d\to\infty$. Then, Eq. (28) with zero left side gives the desired Gaussian PDF. When $\eta>0$ but is small or d is finite but large, one can develop a perturbation expansion around this Gaussian state and evaluate high-order moments of the scalar differences. This has been done in important papers by Gawedzki *et al.* [8] and Chertkov *et al.* [9]. In this work we are interested in the case of $\eta=O(1)$ which is far from a pure diffusive Gaussian limit. Thus, in fact, Eq. (28) is based on an additional assumption that the function $f(r,\eta,d)$ has a universal, η- and d-independent limit and set $f(r,\eta,d)=f(r)$ in Eq. (28). It is clear that the scale-invariant solution of (28) derived below does not tend to a Gaussian in the limits $\eta\to0$ or $d\to\infty$.

The probability density must satisfy the normalization constraint:

$$\int P(T_-,r)dT_-=1. \tag{29}$$

We will be seeking the scale-invariant solution of (28)

$$P(T_-,r)=\frac{1}{\sqrt{S_2(r)}}F(x),$$

where now $x=T_-/\sqrt{S_2(r)}$. It is clear from (29) that

$$\int F(x)dx=1. \tag{30}$$

Since (28) is a nonlinear equation, the PDF is not determined up to an arbitrary constant and the normalization condition serves as a constraint on the coefficients a, b, and c. The equation for the second-order moment $S_2(r)$, obtained from (28), gives

$$\int x^2F(x)dx=1, \tag{31}$$

which provides us with an additional condition:

$$m=a-c-bM=1, \tag{32}$$

where

$$M=\int F^2(x)dx. \tag{33}$$

The equation for $F(x)$ is

$$(2\alpha^2x^2+cd\xi_2)\frac{\partial F(x)}{\partial x}+d\xi_2 b\frac{\partial F^2(x)}{\partial x}+AxF(x)=0. \tag{34}$$

We have an equation with three unknown coefficients which can be found from three constraints introduced above.

The tails of the PDF ($x\rightarrow\infty$) are evaluated readily:

$$F(x)\propto\left(\frac{2\alpha^2x^2}{cd\xi_2}+1\right)^{-A/4\alpha^2}, \tag{35}$$

where $A=2\alpha^2+2(\xi_2-d)\alpha+ad\xi_2$. This result can be obtained directly from the Boltzmann equation (27). One can see that when T_- is large, the account of the higher-order collisions does not change the result. The parameters a, b, and c are functions of both $\xi_2=2\alpha$ and the space dimensionality d. Let us rewrite (34) as

$$\left(\frac{\alpha}{d}x^2+c\right)\frac{\partial F(x)}{\partial x}+b\frac{\partial F^2(x)}{\partial x}+\left(\frac{3\alpha}{d}+a-1\right)xF(x)=0. \tag{34$'$}$$

It is obvious that the most important parameter of the problem is the ratio

$$\gamma=\frac{\alpha}{d}.$$

When $\gamma\rightarrow 0$ the tails of the solution to Eq. (34$'$) decrease somewhat more rapidly than Gaussian, while the central part is a little bit wider than that of the Gaussian distribution. This is the only way to satisfy the constraints (30)–(33) stating that in the units, adopted in this work $S_0=S_2=1$. This means that the expansion introduced in this work does not cover interesting close-to-Gaussian cases of large space dimensionality $d\rightarrow\infty$ and the logarithmic theory with $\xi_2\rightarrow 0$. To remedy this drawback one has to consider a more complicated function $f(r,\eta)$ in Eq. (28).

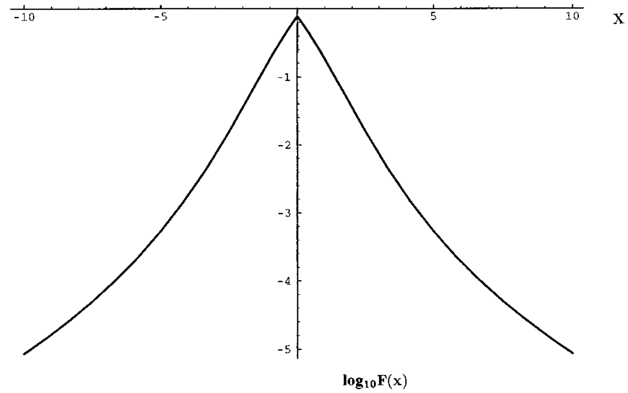

FIG. 1. Probability density $P(\Delta T,r)$ plotted as a function of the scaled variable for the parameters from set I.

In all parameters in (34) are finite, the moments S_n with $n>p=A/4a^2$ are infinite. This does not pose any problem if we recall that the theory developed above is valid only for the scalar fluctuations with $|T_-|\ll T_{rms}$. If $|T_-|>T_{rms}$, the probability of both $T(2)$ and $T(1)\gg T_{rms}$ is small and, as a consequence, the function $P(T_-,r)$ is a sharply decreasing function, dominated by the single-point r-independent probability density $P(T)$. In other words, the tails of the PDF of velocity differences are described by the function

$$P(T_-,r)\approx\psi(r)\varphi\left(\frac{T_-}{T_{rms}}\right), \tag{36}$$

where $\varphi(x)$ is a function x decreasing with x so sharply that all moments are finite. With this assumption the moments S_n with $n>p$ can be evaluated readily with the result:

$$S_n(r)\propto r^{(2p-1)\alpha}, \tag{37}$$

which means that the tail of the PDF, responsible for the behavior of the high-order moments, is given by

$$P(T_-,r)\propto r^{(2p-1)\alpha}\phi\left(\frac{T_-}{T_{rms}}\right) \tag{38}$$

with $\phi(y)$ independent on r. The physical interpretation of this result is simple: Consider $T_-\gg T_{rms}$. The probability of both $|T(2)|\gg T_{rms}$ and $|T(1)|\gg T_{rms}$ is very small. Thus, it is a natural assumption that when $|T_-\gg T_{rms}$ only one of the values of $T(1)$ or $T(2)$ is very large and another one is not. The probability of such event is independent of the value of r and is dominated by the single-point PDF $P(T)$.

Equation (34) has been solved numerically subject to constraints (30)–(33) for the values of parameters used in numerical experiments by Chen [3] $d=2$; $\xi_2=2\alpha=1/2$. The solution for $a=1/1.237+5/8$; $b=-0.322$ and $c=1/2$ (set I) is presented in Figs. 1 and 2. The accuracy of the solution is demonstrated by checking against the constraints derived above:

$$\int_{-200}^{200}F(x)dx=1.066,$$

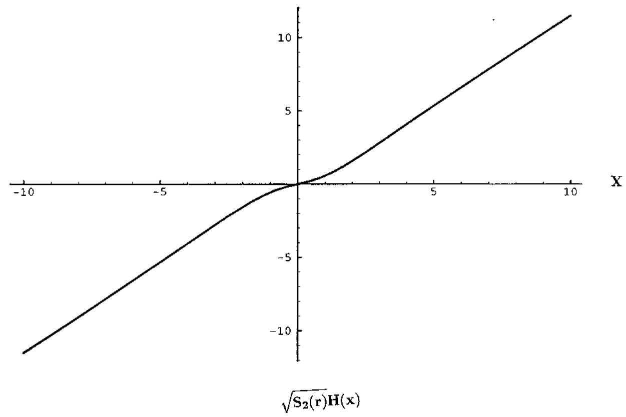

FIG. 2. The function $H(\Delta T,r)$ defined by (39) (set I).

$$\int_{-200}^{200} x^2 F(x)\,dx = 1.02.$$

The value of parameter M evaluated numerically is $M=0.43$ and it is easy to check that for this case $m=1.07$. Since by (32) $m=1$, this result determines the accuracy of the calculation. The value of the scaling function at the origin $F(0)=1/1.290$. It should be mentioned that the solution is extremely sensitive to the value of the coefficients: the 10–20 % modification of the numerical value of one of the coefficients often leads to a few orders of magnitude changes in the result. For these values of parameters the predicted scaling of the moments with $n>p=5.4$ is

$$S_n \propto r^{\xi}$$

with $\xi \approx 5.4/4 \approx 1.35$ and in the scale-invariant range the PDF is

$$P(T_-,r) \propto r^{-1/4} x^{-6.4}.$$

The moments S_n with $0<n<5.4$ obey the simple scaling $S_n \propto \tau^{\xi_n}$ where $\xi_n = n/4$.

The best result satisfying all constraints (30)–(33) was obtained for $a=1/1.8+5/8$, $b=-0.14$, and $c=0.24$ (set II). The results are presented in Figs. 3 and 4. For these values of parameters we have

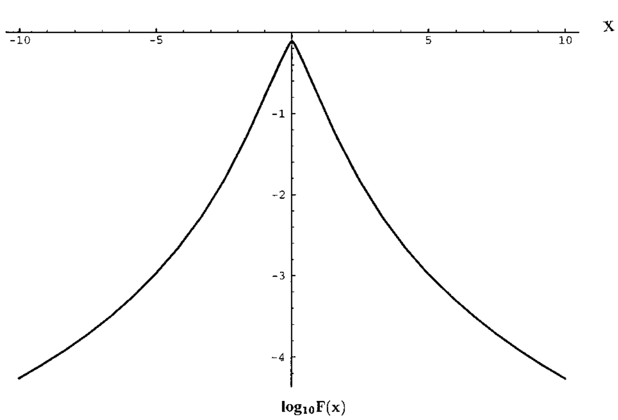

FIG. 3. The same as Fig. 1 for parameters from set II.

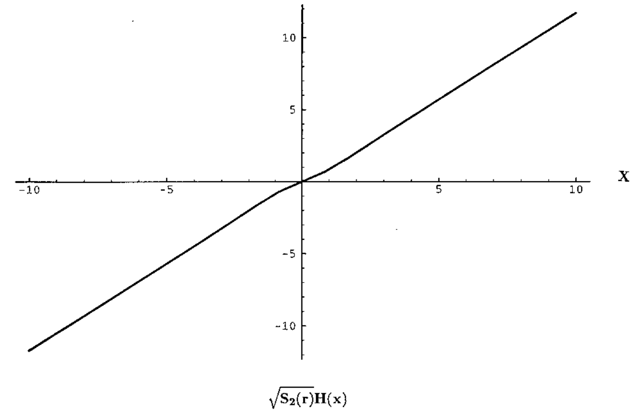

FIG. 4. The same as Fig. 2 for parameters from set II.

$$\int_{-300}^{300} F(x)\,dx = 0.973,$$

$$\int_{-300}^{300} x^2 F(x)\,dx = 1.005,$$

and $m=0.992$. The PDF in the scale-invariant regime is

$$P(T_-,r) \propto r^{-1/4} x^{-4.4}$$

and the moments with $n>p=3.4$ scale with the exponents $\xi_n = 3.4/4 = 0.85$.

As we see, the results are sensitive to the precise values of parameters a, b, and c but it is clear that for the Chen case [3] $\xi_n \approx 0.85-1.4$ and the limiting order of the moment $p \approx 3.4-6.4$.

Comparing Eqs. (9) and (28) we see that

$$-\frac{\sqrt{S_2(r)}}{\xi_2 d} H(T_-) = ax + c\,\frac{\partial \ln(F)}{\partial x} + 2b\,\frac{\partial F}{\partial x}. \qquad (39)$$

When x is large this expression is similar to Kraichnan's conjecture (10) with the different value of the coefficient. In the region of small $x \ll 1$ the expression for $H(x)$, derived in this work, strongly deviates from (10), enabling the existence of the scale-invariant PDF. It should be stressed that all numerical factors entering the equations of motion and the expression for $H(x)$ strongly depend on both space dimensionality d and the exponent of the velocity structure function ξ_2. It is clear from the derivation that expression (39) is valid only when both $\tau/L \ll 1$ and $T_- \ll T_{rms}$ and, as a consequence, for any finite value of τ/L it fails at $x \approx T_{rms}/\tau^{\alpha}$. This prediction can be easily checked numerically.

This work makes some predictions which can be verified numerically. In the region of small $x \ll 1$ and $T_- \ll T_{rms}$ all curves

$$F(x) = r^{\alpha} P(T_-/r^{\alpha})$$

should collapse onto the same curve for any value of the displacement r from the inertial range. The tails of these curves strongly deviate from each other. Qualitative behavior

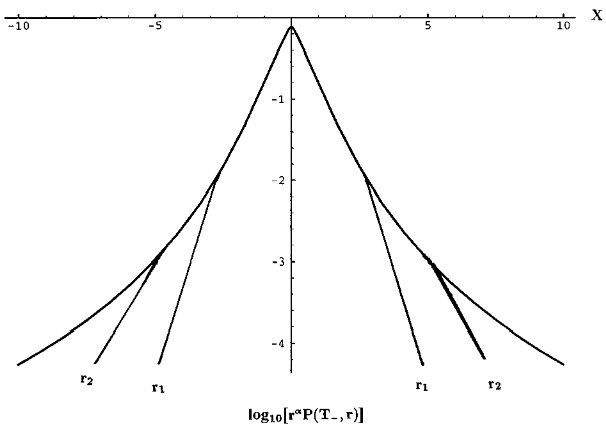

FIG. 5. The function $r^{\alpha}P(T_-,r)$ as a function of the separation distance r. (Schematic: $r_1>r_2$.)

of the PDF's plotted as a function of a scaled variable is presented in Fig. 5. On the other hand, plotted as a function of the r-independent variable

$$\frac{1}{r^{(2p-1)\alpha}}\,P\left(\frac{T_-}{T_{rms}},r\right)$$

all PDF's should collapse in the region $T_-\gg T_{rms}$ and strongly deviate from each other near the origin $T_-\ll T_{rms}$. The value of parameter $p=A/4\alpha^2$, which determines the exponents of all high-order structure functions depends on the parameters of the problem.

To conclude this section let us discuss the role of the approximations involved. Equation (28) was obtained neglecting the multiple collisions in the collision integral (27). It is easy to see that this approximation does not effect the shape of the tails of the PDF since there $\delta^{n-1}P^n(T_-)\ll P^2(T_-)$ for all $n>2$, where δ is the width of the interval contributing to the corresponding collision integral. Also, deriving (28) the binary collision term in (27) was replaced by $O(P^2)$ contribution to (28) which is correct only if the solution is broad enough near the origin. This assumption has been tested by substitution of the numerical solution to (28) into the binary collision integral in (27). In the interval $0<x<1$ the resulting expression was very close to its approximation used in derivation of (28). A similar conclusion can be reached regarding the last term in (26). In principle, the symmetries of the problem allow some other expressions involving higher powers of $f(r)(\partial/\partial\lambda)$. In the scale-invariant regime in which we are interested, these terms lead to the high-order derivatives $\partial^m F(x)/\partial x^m$, which are small when χ is large. Thus, these terms, if they do exist, cannot modify the algebraic tails of the PDF $F(x)$.

IV. SUMMARY AND DISCUSSION

The theories of homogeneous and isotropic forced Navier-Stokes turbulence are invariably based on the Kolmogorov assumption that all symmetries of the unforced Euler equations, including Galileo invariance, are restored in the inertial range where separations between the points are small ($r/L\ll 1$). This assumption introduced an important dynamical constraint on possible classes of solutions of the problem and led to various field-theoretical approaches, giving reasonably accurate predictions of the low-order statistics in turbulent flows, while completely failing to describe experimentally observed intemittency dominating high-order moments of the velocity differences. In a recent work on the Burgers turbulence Polyakov [4] introduced a new principle: the symmetries of the free, unforced, equation of motion are restored only when both $r/L\ll 1$ and $|u(2)-u(1)|\ll u_{rms}$. If $|u(2)-u(1)|>u_{rms}$, the dissipation terms couple Δu and $u(2)+u(1)$, thus leading to violation of Galileo invariance of the unforced equation of motion. This statement is easy to understand since for very large values $|u(2)-u(1)|\approx|u(2)+u(1)|$ with high probability. In this range of variation of the variable the PDF scales with u_{rms} which is not Galileo invariant. The Burgers turbulence is dominated by the shocks and one can see that this assumption is naturally satisfied there.

A solution of the problem of a passive scalar, presented in this work, is based on a similar assumption: only when $|T_-|\ll T_{rms}$ and $r/L\ll 1$ the invariance of the PDF $P(T_-,r)$ under the transformation $T\to T+C$, where C is a constant, is restored. If $|T_-|\gg T_{rms}$, the PDF is dominated by a single-point, r-independent probability density which is not invariant under constant shift of the variable T. The second assumption about scale-invariant shape of the PDF, essential for the presented solution, enforces the ''normal scaling'' of the low-order moments and is consistent with the equations of motion. The derived algebraically decreasing PDF explains strong intermittency and ''anomalous scaling'' of high-order moments. However, at the present time we do not have a solid theoretical ground to justify the assumption about scale-invariance of the PDF in the vicinity of the origin $|T_-|\ll T_{rms}$ and one cannot rule out another scenario. For example, the $O((r/L)^2)$ contribution from the forcing function has been neglected in the above derivation of PDF. In principle, it is not impossible that for $S_n(r)$ with $n\to 0$ this term becomes important, generating a crossover to another scaling of the lowest-order moments. This possibility, which is somewhat bizarre, cannot be ruled out until all assumptions involved in the derivation are verified. The numerical experiments can be of great help here. I would like to emphasize the paramount importance of extremely accurate investigation of the PDF $P(T_-,r)$ in the vicinity of the top where $T_-\to 0$. This suggestion is in contrast with a typical experimental emphasis on the properties of the tails responsible for anomalous scaling.

To conclude, I would like to discuss the recent theories of a passive scalar by Gawedzki *et al.* [8] and Chertkov *et al.* [9]. Both papers present systematic expansions around Gaussian solutions which are possible in two limiting cases $d\to\infty$ [9] and $\xi_2=2$ [8]. Both works have demonstrated the anomalous scaling of the fourth-order structure functions which is inconsistent with the normal scaling predicted in this work. It is important to understand the reasons for the differences. Chertkov *et al.* [9] have considered an equation for general four-point correlation function in the limit of large space dimensionality: $d\to\infty$. Using $1/d$ as a small parameter, they have shown that the four-point correlation function $F_4=\langle T(1)T(2)T(3)T(4)\rangle$ scales under the transformation $r\to ar$ as $F_4'\propto a^{2\xi_2}[1+O(\Delta_4\ (\log(a)/d))]F_4$ where $\Delta_4=O(1)$. Then they concluded that, in fact, this relation is a first term of an expansion of the exponential, leading to the anomalous scaling exponent of the fourth-order correlation function equal to

$$\overline{\xi_4} = 2\xi_2 - \frac{\Delta_4}{d} + O(1/d^2)$$

The $O(1/d^2)$ contribution has been neglected in the limit $d \to \infty$, giving the nontrivial scaling exponent of the four-point correlation function. The result obtained in [9] is valid only when d is large enough so that $n \ll d$. Gawedzkii and Kupiainen [8] considered a case $\eta \to 0$ and derived an anomalous scaling exponent similar to that of Ref. [9]. Their perturbation expansion is valid when $n(n-2)n/d \ll 1$. Thus, the results of Refs. [8] and [9] cannot be used to evaluate the scaling exponents of the high-order structure functions when d is not too large or η is not too small. The theory presented in this paper assumes that far from a Gaussian limit there exist a range of parameters for which a universal relation $S_0(x) = S_2(x) = 1$ where $x = T_- / \sqrt{S_2(r)}$ holds. If this is so, then the scale-invariant PDF of the scalar differences is possible in the range $T_- \ll T_{rms}$ and $r/L \ll 1$.

It is tempting to relate the limiting exponents of the structure functions to geometrical characteristics of scalar field. We know [4–7] that the driven Burgers equation gives all $S_n \propto r$ for $n \gg 1$. The explanation of this fact is straightforward: the dominant contribution to the structure functions comes from strong coherent shocks, which is the most prominent dynamical feature of the system. In the scalar theory, developed in this paper, the value of the limiting exponents depends on both space dimensionality and power of the velocity spectrum. The possibility of existence of the large-scale structures in the scalar field, dominating the high-order moments remains an open and interesting question.

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